# From Reissner-Nordström quantum states to charged black holes mass evaporation

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In this report we describe quantum Reissner-Nordström (RN) black-holes interacting with a complex scalar field. Our analysis is characterized by solving a Wheeler-DeWitt equation in the proximity of an apparent horizon of the RN space-time. Subsequently, we obtain a wave-function $\Psi_{\rm RN}[M,Q]$ representing the RN black-hole. A special emphasis is given to the evolution of the mass-charge rate affected by Hawking radiation. More details can be found in ref. [12].

Recently, there has been a renewed interest in the canonical quantization of black-hole space-times [1–12]. The general aim is to obtain a description of quantum black holes that would go beyond a semi-classical approximation. In this report, we will extend M. Pollock's method [1] (which was itself influenced by the work of A. Tomimatsu [4]) to Reissner-Nordström (RN) black-hole. As a consequence, we will find a wave function for the RN black-hole, which will have an explicit dependence on its mass $M$ and the charge $Q$. The T.O.H. (Tomimatsu-Oda-Hosoya) method[5,6] constitutes another very interesting and similar approach to this purpose.

The main elements of the Hamiltonian formalism used here follow ref. [13]. En route towards our reduced model we further take the following steps. To begin with, our 4-dimesional *spherically symmetric* metric is written as

$$ds^2 = h_{ab}dx^a dx^b + \phi^2(d\theta^2 + \sin^2\theta d\varphi^2), \tag{1}$$

together with the ADM decomposition

$$h_{ab} = \begin{pmatrix} -\alpha^2 + \frac{\beta^2}{\gamma} & \beta \\ \beta & \gamma \end{pmatrix}, \; h^{ab} = \begin{pmatrix} -\frac{1}{\alpha^2} & \frac{\beta}{\alpha^2\gamma} \\ \frac{\beta}{\alpha^2\gamma} & \frac{1}{\alpha} - \frac{\beta^2}{\alpha^2\gamma^2} \end{pmatrix}, \tag{2}$$

where $\alpha, \beta, \gamma, \phi$ are functions of $(x^0, x^1) = (\tau, r)$ which will be defined later (see eq. (3)). We also take a scalar field $\hat{\psi} = \psi(\tau, r)$ and the electromagnetic field $F_{ab} = \varepsilon_{ab}\sqrt{-h}E$; $E = (-h)^{-1/2}(\dot{A}_1 - A_0')$, $\hat{D}_a\hat{\psi} = \partial_a\psi + ieA_a\psi$; $\hat{D}_2\hat{\psi} = \hat{D}_3\hat{\psi} = 0$. Notice we employ "$\cdot$" $\equiv \frac{\partial}{\partial\tau}$ and "$\prime$" $\equiv \frac{\partial}{\partial r}$. The next *significant* step consists in choosing the following coordinate gauge: $\alpha = \frac{1}{\sqrt{\gamma}} \Leftrightarrow \sqrt{-h} = 1$.

The constraints of our model will be expressed in terms of dynamical quantities defined at an apparent horizon of the RN black-hole, which is defined by the condition[13]: $h^{ab}(\partial_a\phi)(\partial_b\phi) = 0 \Leftrightarrow \dot{\phi}(\dot{\phi} - \phi') = 0$. In order to obtain a satisfactory description of an evaporating RN black hole on the apparent horizon, it is more convenient to use a RN - Vaidya metric [1]. With $v \equiv \tau + r = t + r^*$ as the advanced

null Eddington-Finkelstein coordinate, and $r^*$ as the corresponding "tortoise" coordinate, the relationship between the time $\tau$ and the time coordinate $t$ for the standard RN metric is $\tau = t - r + r^*$. The Vaidya RN metric becames

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) d\tau^2 + \left(\frac{4M}{r} - \frac{2Q^2}{r^2}\right) d\tau dr + \left(1 + \frac{2M}{r} - \frac{Q^2}{r^2}\right) dr^2 + \phi^2 d\Omega_2^2. \tag{3}$$

The RN black-hole has two apparent horizons, namely at $r_\pm(v) = M(v) \pm \sqrt{M^2(v) - Q^2}$ and we will henceforth restrict ourselves to the case of $r_+$. In similarity with the Schwarzschild case we also take $M \simeq M(\tau)$ in the vicinity of the apparent horizon. After some lenghty calculations we obtain (see ref. [12] for more details) the approximate expressions:

$$\begin{aligned} \mathcal{H}_0 &= \frac{1}{2}\pi_\phi - \frac{1}{8} + \frac{Q^2}{32M^2} - \frac{1}{16\pi}\frac{1}{\rho^2}(\pi_\chi^2 + \pi_{\psi_1}^2) - \rho^2\pi({\psi_1'}^2 + \chi'^2) \\ &- \frac{1}{2\rho^2}\pi_{A_1}^2 - \frac{\rho^2}{2}[2\pi e^2 A_1^2(\psi_1^2 + \chi^2)] - 2\pi e\rho^2 A_1(\psi_1\chi' - \chi\psi_1')] \end{aligned} \tag{4}$$

$$\mathcal{H}_1 = \frac{1}{2}\pi_\phi - \frac{1}{8} - \frac{3Q^2}{32M^2} + \frac{1}{2}[\pi_{\psi_1}\psi_1' + \pi_\chi\chi'] + \frac{eA_1}{4}(\pi_{\psi_1}\chi - \pi_\chi\psi_1), \tag{5}$$

with $\psi = \sqrt{2\pi}(\psi_1 + i\chi), \pi_\psi \to \frac{1}{\sqrt{8\pi}}\pi_\psi$. Compatibility[12] requires that $\pi_{\psi_1} = -4\pi\rho^2\psi_1'$, $\pi_\chi = -4\pi\rho^2\chi'$, with the terms with $A_1$ to be negligible, and with the conditions $A_0 = 0$ together with $\dot\psi = \dot\psi^\dagger = 0$. We further take $A_1 = A_1(r)$.

Quantization proceeds via the operator replacements

$$\pi_\phi \to -i\frac{\partial}{\partial\phi} \simeq -i\frac{2M^2}{4M^2 + Q^2}\frac{\partial}{\partial M}, \pi_{\psi_1} \to -i\frac{\partial}{\partial\psi_1}, \pi_\chi \to -i\frac{\partial}{\partial\chi}, \tag{6}$$

which yelds the Wheeler-DeWitt equation for the wave function $\Psi$,

$$-i\frac{2M^2}{4M^2 + Q^2}\frac{\partial\Psi}{\partial M} = -\frac{1}{16\pi M^2 + \pi Q^4/M^2 - 8\pi Q^2}\left[\frac{\partial^2\Psi}{\partial\psi_1^2} + \frac{\partial^2\Psi}{\partial\chi^2}\right] + \frac{1}{4}\Psi, \tag{7}$$

whose solutions are

$$\Psi_{\rm RN}[M, Q; \psi_1, \chi; k] = \Psi_{\rm RN}^0 e^{i\left[\frac{1}{4}\left(-\frac{Q}{M} + 2M\right) + k^2\frac{M}{Q^2 - M^2} \pm 2\sqrt{\pi}k(\psi_1 + \chi)\right]}\zeta(A_1), \tag{8}$$

where $k^2$ is a separation constant and $\Psi_{\rm RN}^0$ an integration constant.

It is interesting to notice the following as well. For the Schwarzschild case ($Q = 0$) eq. (8) implies that near to $M = 0$ the wave function will oscilate with infinite frequency. If $\dot M < 0$, this would represent the quantum mechanical behaviour of the black hole near the end point of its evaporation. In the RN case, the rapid oscillations will occur again for $M = 0$ but also when $M \sim Q$. I.e., near extremality and when the black hole mass evaporation can eventually stop. Hence, the presence of $Q$ in $\Psi_{\rm RN}$ allowed us to identify some known physical situations of the RN bkack hole.

As far as the mass-charge ratio for the RN black hole is concerned, we get the equation

$$\dot{M} + \frac{1}{4M^2}\frac{a[k^2;Q]}{d(M)} + \frac{1}{4M^4}\frac{b[k^2;Q]}{d(M)} + \frac{c[k^2;Q]}{d(M)M^6} = 0, \tag{9}$$

where

$$a[k^2;Q] = k^2 - 5Q^2 + Q; b[k^2;Q] = k^2Q^2 - 3Q^4; c[k^2;Q] = \frac{k^2Q^4}{8}; d(M) = 1 + \frac{Q^2}{2M^2}. \tag{10}$$

An integration of (9) leads to the result

$$M = \left[M_0^3 - \frac{3}{2}(k^2 - 5Q^2 + Q)\right]^{1/3} (t - t_0)^{1/3}. \tag{11}$$

We can now identify several physical cases of interest for the RN black hole, according if $a, b, c, d$ are either positive, zero or negative.

For the case of $Q = 0$ [1] (Schwarzschild), it is the separation constant $k^2$ that determines if the black hole is evaporating and decreasing its mass ($k^2 > 0$), or increasing its mass ($k^2 < 0 \Leftrightarrow k$ imaginary). In the present RN black hole case, the presence of the charge $Q$ introduces significative changes. A $\dot{M} > 0$ stage can be obtained, with $k^2 > 0$ but $a < 0, b < 0$ and $c > 0, d > 0$. If $Q = 0$, this possibility is absent. When $Q \neq 0$, $k^2 < 0$, then $c < 0, b < 0, a < 0$, if $1 + 20k^2 > 0$ has real solutions.

Overall, our results do bring additional information regarding quantum black-hole, but there is still the need for further investigation.

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